Université Abdelmalek Es-saadi
FST Tanger

Année 2008-09
M112 : G6 & GE-GM

[1] Contrôle continu N^0 : 2

Exercice 1: Résoudre les équations suivantes :

a) $2y'' - 7y' + 3y = x^2 - 2e^{2x}$

b) $y' \sin x + y \cos x = 1$

Exercice 2:

a) Trouver les valeurs a, b, c et d telles que :

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+x\sqrt{2}+1} + \frac{cx+d}{x^2-x\sqrt{2}+1}$$

b) Calculer $I = \int_0^1 \frac{1}{x^4+1} dx$

c) Déduire

i) $J = \int_0^1 \frac{1}{(x^4+1)^2} dx$

ii) $K = \int_0^1 \frac{x+1}{(x^4+1)^2} dx$.

Exercice 3: Calculer les limites suivantes en utilisant le DL

a) $\lim_{x \to 0} \frac{1}{x} - \frac{1}{\ln(1+x)}$ b) $\lim_{x \to +\infty} x^2[\ln(x+\sqrt{1+x^2}) - \ln x]$

Exercice 4: Soit la suite récurente $(u_n)_n$ définie par :

$$\begin{cases} u_n := \ln(1+u_{n-1}) \\ u_0 > 0 \end{cases}$$

a) Montrer que la suite $(u_n)_n$ est strictement décroissante

b) Déduire que la suite $(u_n)_n$ est une suite convergente et déterminer sa limite

Institut la centrale CC2 08-09

Exercice 1 a/ (E) : $2y''-7y'+3y = x^2-2e^{2x}$

. Resolution de (E') : $2y''-7y'+3y=0$

l'eq caracteristique associée est : $2r^2-7r+3=0$ $\Delta = 25$, $r_1=1/2$, $r_2=3$

$$y_1 = \alpha e^{x/2}+\beta e^{3x} \text{ sol de } (E')$$

. Solution particulière y_0 de (E) s'ecrit $y_0=y_{01}+y_{02}$ avec y_{01} sol particulière de l'eq (E_1) : $2y''-7y'+3y=x^2$ et y_{02} sol particulière de (E_2) : $2y''-7y'+3y=-2e^{2x}$

. $y_{01}=ax^2+bx+c$ car $\deg(x^2)=2$ et $c\neq 0$

$y_{01}'=2ax+b$, $y_{01}''=2a$

Alors $2y_{01}''-7y_{01}'+3y_{01}=x^2 \Rightarrow 4a-14ax-7b+3ax^2+3bx+3c=x^2 \Rightarrow \begin{cases}3a=1\\-14a+3b=0\\4a-7b+3c=0\end{cases}$

$\Rightarrow a=1/3$; $b=\frac{14}{9}$; $c=\frac{86}{27}$ $\Rightarrow y_{01}=\frac{1}{3}x^2+\frac{14}{9}x+\frac{86}{27}$

. $y_{02}=ae^{2x}$ car $m=2$ n'est pas racine de l'eq caracteristique

$y_{02}'=2ae^{2x}$, $y_{02}''=4ae^{2x}$

Alors $2y_{02}''-7y_{02}'+3y_{02}=-2e^{2x}$; $8ae^{2x}-14ae^{2x}+3ae^{2x}=-2e^{2x} \Rightarrow -3a=-2 \Rightarrow a=\frac{2}{3}$

$\Rightarrow y_{02}=2/3\, e^{2x}$ $\Rightarrow y_0=y_{01}+y_{02}=\frac{1}{3}x^2+\frac{14}{9}x+\frac{86}{27}+\frac{2}{3}e^{2x}$

et $y=y_1+y_0=\alpha e^{x/2}+\beta e^{3x}+\frac{1}{3}x^2+\frac{14}{9}x+\frac{86}{27}+\frac{2}{3}e^{2x}$ sol de E

b/ (E) : $y'\sin x+y\cos x=1$

. (E') : $y'\sin x+y\cos x=0 \to \frac{y'}{y}=-\frac{\cos x}{\sin x}$; $\ln|y|=-\ln|\sin x|+C \Rightarrow y_1=\frac{K}{\sin x}$

. Determinons une solution particulière de E sous la forme $y_0=\frac{K}{\sin x}$, avec K variable

$y_0'=\frac{K'\sin x-K\cos x}{\sin^2 x}$; Alors $y_0'\sin x+y_0\cos x=1 \Rightarrow \frac{K'\sin x-K\cos x}{\sin x}+\frac{K\cos x}{\sin x}=1$

$\Rightarrow K'=1 \Rightarrow K=x$ et $y_0=\frac{x}{\sin x}$ d'où $y=y_1+y_0=\frac{K+x}{\sin x}$ sol de (E)

Exercice 2 a/ $\frac{1}{x^4+1}=\frac{ax+b}{x^2+x\sqrt{2}+1}+\frac{cx+d}{x^2-x\sqrt{2}+1} \Rightarrow 1=(ax+b)(x^2-x\sqrt{2}+1)+(x^2+x\sqrt{2}+1)(cx+d)$

$(a+c)x^3+(-a\sqrt{2}+b+c\sqrt{2}+d)x^2+(a-b\sqrt{2}+c+d\sqrt{2})x+b+d=1 \Rightarrow \begin{cases}a+c=0\\-a\sqrt{2}+b+c\sqrt{2}+d=0\\a-b\sqrt{2}+c+d\sqrt{2}=0\\b+d=1\end{cases}$

$\begin{cases}c=-a\\d=1-b\\-a\sqrt{2}+b-a\sqrt{2}-b+1=0\\a-b\sqrt{2}-a-b\sqrt{2}+\sqrt{2}=0\end{cases} \Rightarrow \begin{cases}a=\sqrt{2}/4\\b=1/2\\c=-\sqrt{2}/4\\d=1/2\end{cases}$

b/ $\frac{1}{x^4+1}=\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+x\sqrt{2}+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-x\sqrt{2}+1}=\frac{\sqrt{2}}{4}\left(\frac{x+\sqrt{2}}{x^2+x\sqrt{2}+1}-\frac{x-\sqrt{2}}{x^2-x\sqrt{2}+1}\right)$

$=\frac{\sqrt{2}}{8}\left(\frac{2x+2\sqrt{2}}{x^2+x\sqrt{2}+1}-\frac{2x-2\sqrt{2}}{x^2-x\sqrt{2}+1}\right)=\frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}+\sqrt{2}}{x^2+x\sqrt{2}+1}-\frac{2x-\sqrt{2}-\sqrt{2}}{x^2-x\sqrt{2}+1}\right)$

$$\frac{1}{x^4+1} = \frac{\sqrt{2}}{8}\left(\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1} + \frac{\sqrt{2}}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} - \frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1} + \frac{\sqrt{2}}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)$$

$$F(x) = \frac{\sqrt{2}}{8}\left(\ln(x^2+x\sqrt{2}+1) + \sqrt{2}.\sqrt{2}.\operatorname{Arctan}\left(\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right) - \ln(x^2-x\sqrt{2}+1) + \sqrt{2}.\sqrt{2}\operatorname{Arctan}\left(\frac{x-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)\right)$$

$$= \frac{\sqrt{2}}{8}\left(\ln\left(\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right) + 2\operatorname{Arctan}(\sqrt{2}x+1) + 2\operatorname{Arctan}(\sqrt{2}x-1)\right)$$

$$\int_0^1 \frac{1}{x^4+1}\,dx = [F(x)]_0^1 = F(1)-F(0) = \frac{\sqrt{2}}{8}\left(\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + 2\operatorname{Arctan}(1+\sqrt{2}) + 2\operatorname{Arctan}(\sqrt{2}-1)\right)$$

c/ i/ $I = \int_0^1 \frac{1}{x^4+1}\,dx$ On pose $\begin{cases} u = \frac{1}{x^4+1} = (x^4+1)^{-1} \\ v' = 1 \end{cases}$ $\begin{cases} u' = -4x^3(x^4+1)^{-2} \\ v = x \end{cases}$

$$I = \left[\frac{x}{x^4+1}\right]_0^1 - \int_0^1 \frac{-4x^4}{(x^4+1)^2}\,dx = \frac{1}{2} + 4\int_0^1 \frac{x^4}{(x^4+1)^2}\,dx = \frac{1}{2} + 4\int_0^1 \frac{x^4+1-1}{(x^4+1)^2}\,dx$$

$$I = \frac{1}{2} + 4\left(\int_0^1 \frac{x^4+1}{(x^4+1)^2}\,dx - \int_0^1 \frac{1}{(x^4+1)^2}\,dx\right) \Rightarrow I = \frac{1}{2} + 4I - 4J \Rightarrow 4J = 3I + \frac{1}{2}$$

$$J = \frac{3}{4}I + \frac{1}{8} = \frac{3\sqrt{2}}{32}\left(\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + 2\operatorname{Arctan}(1+\sqrt{2}) + 2\operatorname{Arctan}(\sqrt{2}-1)\right) + \frac{1}{8}$$

ii/ $K = \int_0^1 \frac{x+1}{(x^4+1)^2}\,dx = \int_0^1 \frac{x}{(x^4+1)^2}\,dx + \int_0^1 \frac{1}{(x^4+1)^2}\,dx = L + J$

$L = \int_0^1 \frac{x}{(x^4+1)^2}\,dx = \int_0^1 \frac{1/2\,dt}{(t^2+1)^2} = \frac{1}{2}\int_0^1 \frac{dt}{(t^2+1)^2}$ On pose : $x^2 = t$; $2x\,dx = dt$

De plus $\int_0^1 \frac{1}{t^2+1}\,dt = [\operatorname{Arctan} t]_0^1 = \pi/4$

et $\int_0^1 \frac{1}{t^2+1}\,dt = \left[\frac{t}{t^2+1}\right]_0^1 - \int_0^1 \frac{-2t^2}{(1+t^2)^2}\,dt$ $\begin{cases} u = \frac{1}{t^2+1} = (t^2+1)^{-1} \\ v' = 1 \end{cases}$ $\begin{cases} u' = -2t(t^2+1)^{-2} \\ v = t \end{cases}$

$$\frac{\pi}{4} = \frac{1}{2} + 2\int_0^1 \frac{t^2}{(1+t^2)^2}\,dt = \frac{1}{2} + 2\left(\int_0^1 \frac{t^2+1-1}{(1+t^2)^2}\,dt\right) = \frac{1}{2} + 2\left(\int_0^1 \frac{1}{1+t^2}\,dt - \int_0^1 \frac{1}{(1+t^2)^2}\,dt\right)$$

$$\frac{\pi}{4} = \frac{1}{2} + 2\left(\frac{\pi}{4} - 2L\right) \Rightarrow \frac{\pi}{4} = 1/2 + \pi/2 - 4L \Rightarrow L = \frac{\pi+2}{16}$$

$$\text{d'où } K = \frac{\pi+2}{16} + \frac{3\sqrt{2}}{32}\left(\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + 2\operatorname{Arctan}(1+\sqrt{2}) + 2\operatorname{Arctan}(\sqrt{2}-1)\right) + \frac{1}{8}$$

Exercice 3 a/ $\frac{1}{x} - \frac{1}{\ln(1+x)} = \frac{\ln(1+x)-x}{x\ln(1+x)} = \frac{x - \frac{x^2}{2} + x^2\varepsilon(x) - x}{x(x - \frac{x^2}{2} + x^2\varepsilon(x))} = \frac{-\frac{1}{2} + \varepsilon(x)}{1 + \frac{x}{2}\varepsilon(x)} \xrightarrow[x\to 0]{} -1/2$

$$x^2(\ln(x+\sqrt{1+x^2}) - \ln x) = x^2\left(\ln x\left(1 + \frac{\sqrt{1+x^2}}{x}\right) - \ln x\right) = x^2\left(\ln x + \ln\left(1 + \frac{\sqrt{1+x^2}}{x}\right) - \ln x\right)$$

$$= x^2\ln\left(1 + \frac{\sqrt{1+x^2}}{x}\right)$$

Posons $X = \frac{1}{x}$ avec $x > 0$ alors $f(x) = \frac{1}{X^2}\ln\left(1 + \frac{\sqrt{1+\frac{1}{X^2}}}{\frac{1}{X}}\right) = \frac{1}{X^2}\ln(1+\sqrt{1+X^2})$

$$\sqrt{1+X^2} = (1+X^2)^{1/2} = 1 + 1/2\,X^2 + X^2\varepsilon(X)$$

$$f(x) = \frac{1}{X^2}\ln(2 + 1/2\,X^2 + X^2\varepsilon(X)) = \frac{1}{X^2}\left(\ln 2 + \ln\left(1 + \frac{1}{4}X^2 + X^2\varepsilon(X)\right)\right)$$

$$= \frac{1}{X^2}\left(\ln 2 + \frac{1}{4}X^2 + X^2\varepsilon(X)\right) = \frac{\ln 2}{X^2} + \frac{1}{4} + \varepsilon(X) = \frac{1}{4} + x^2\ln 2 + \varepsilon\left(\frac{1}{x}\right)$$

$$\lim_{+\infty} f(x) = +\infty$$